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Dissipative tunnelling of the inverted Caldirola–Kanai oscillator

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Abstract. We discuss, in the phase time approach, quantum tunnelling in the presence of dissipation for an inverted oscillator with Caldirola-Kanai damping. The exact expressions of time delay, traversal time and effective tunnelling velocity are derived. Some paradoxical aspects of tunnelling related to the particle speed in crossing the barrier—such as the Hartmann–Fletcher effect—are briefly considered.

1. Introduction

As is well known, there are essentially two different ways of introducing dissipation—microscopically and phenomenologically. To the former category belongs the microscopic model, which has been considered at length by Caldeira and Leggett [1], with the use of path-integral methods in the study of the dissipative quantum tunnelling effect. On the other hand, several phenomenological models of dissipative systems have been proposed, such as the Caldirola–Kanai (CK) equation [2], the Schrödinger–Langevin equation [3], the Gisin equation [4] and the Schrödinger equation with a complex potential [5]. All these models, except for the CK one, have also been used to describe the quantum tunnelling effect in the presence of dissipation [6], a problem that has received a great deal of attention in recent years [7].

In previous works we have introduced a phenomenological model, called the inverted CK equation, that is formally obtained by the standard CK one by the replacement

$$\omega \to i\omega \tag{1}$$

i.e. with a Hamiltonian of the form [8, 9]

$$H=\frac{p^2}{2m}e^{-\gamma t}-\frac{m}{2}\omega^2q^2e^{\gamma t} \tag{2}$$

and studied its behaviour with respect to dissipative quantum tunnelling [10]. The case of a driven, inverted CK model has also been considered [11]. The physics of the CK

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model and of its inverted form (2) are very different: the energy eigenstates of (2) are no longer square-integrable, and they are degenerate, e.g. with respect to incidence from left or right, or, alternatively, with respect to parity. For $\gamma \to 0$ we obviously get the Hamiltonian of the inverted harmonic oscillator which, besides its applications in masers [12], has also been used in reactive scattering [13], due to the fact that for this kind of potential the tunnelling time does not diverge as in the case of a square barrier at threshold in the semiclassical limit analysis [14].

Moreover, two of us (SB and AJ) have shown in [8] that the inverted CK Hamiltonian produces both coherent and squeezed states. As is well known, coherent states represent a unique tool to solve a wide class of problems in the more disparate fields, ranging from quantum optics to superconductivity and even to elementary particles [15]. Indeed, the physical relevance of the inverted CK equation just comes from its combining the main features of the inverted harmonic oscillator with the CK damping. Although it is now widely believed that the CK model does actually represent an oscillator with variable mass and frequency, rather than a genuine damped one [16], it still constitutes a prototype Hamiltonian for an open quantum system, able, for example, to describe the generation of squeezed states out of coherent states by external changes [17]. Therefore, the inverted CK equation may provide, in our opinion, a useful setting for a phenomenological description of quantum tunnelling for systems coupled to an environment.

In [10, 11], assuming as an initial state a Gaussian wave packet, we found the solution of the time-dependent Schrödinger equation for the inverted CK Hamiltonian, and derived the expression of the sojourn (or dwell) time [18, 19]†.

In the present paper, aimed at continuing our study of dissipative quantum tunnelling for the inverted CK model, we shall consider the problem by the phase time approach, essentially based on the time delay [14, 20], which is well known to be connected with the relevant phase shift Φ, according to the relation

$$\tau_d = \hbar \frac{d\Phi}{dE} \tag{3}$$

The paper is organized as follows. In section 2 we solve the Schrödinger equation for Hamiltonian (2) by means of a contact transformation, and find its asymptotic solutions. The time delay, the traversal time and the effective tunnelling velocity are derived in section 3. Concluding remarks, concerning some paradoxical aspects of the tunnelling process, are given in section 4.

2. Solution of the inverted CK equation

First of all, let us solve the time-dependent Schrödinger equation for Hamiltonian (2) in the coordinate representation, which reads

$$i\hbar \frac{\partial \psi(q,t)}{\partial t} = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2} e^{-\gamma t} - \frac{m}{2}\omega^2 q^2 e^{\gamma t}\right)\psi(q,t). \tag{4}$$

By using the contact transformation [21]

† In the following, we shall adopt the two more popular and well established definitions of tunnelling time, those of dwell time and of delay time, ignoring the many contradictions in the various approaches to the problem (the reader is referred to [14–20]).

$$Q=\exp\left(\frac{\gamma}{2}t\right)q \tag{5}$$

and following Dodonov and Man'ko [22], the solution of (4) is given by

$$\psi(Q,t)=\exp\left(-\mathrm{i}\frac{\gamma m}{4\hbar}Q^2-\frac{\mathrm{i}t}{\hbar}E+\frac{\gamma t}{4}\right)f(Q) \tag{6}$$

where the function $f(Q)$ satisfies the following equation:

$$\frac{\hbar^2}{2m}\frac{\partial^2 f}{\partial Q^2}+\left(E+\frac{m}{2}\Omega^2Q^2\right)f(Q)=0 \tag{7}$$

with

$$\Omega=\left(\omega^2+\frac{\gamma^2}{4}\right)^{1/2}. \tag{8}$$

Therefore, the solution of (4) can be written as

$$\psi(q,t)=\exp\left(-\mathrm{i}\frac{\gamma m}{4\hbar}q^2\exp(\gamma t)-\frac{\mathrm{i}tE}{\hbar}+\frac{\gamma t}{4}\right)f\left[q\exp\left(\frac{\gamma t}{2}\right)\right] \tag{9}$$

where $f[q\exp(\gamma t/2)]$ is the solution of (7).

By setting

$$y=\left(\frac{2m\Omega}{\hbar}\right)^{1/2}Q \tag{10}$$

and

$$\mathscr{E}=\frac{E}{\hbar\Omega} \tag{11}$$

(7) takes the form

$$\frac{\partial^2 f(\mathscr{E},y)}{\partial y^2}+\left(\mathscr{E}+\frac{y^2}{4}\right)f(\mathscr{E},y)=0. \tag{12}$$

Equations of the form (7) and (12) have been studied in detail by Barton [20] and Ford *et al* [23] in the q-representation, and by Balasz and Voros [24] in phase space. Their solutions can be expressed in terms of parabolic cylinder (or Weber) functions. In particular, the energy eigenfunctions representing particles incident from the left are given by [20, 23]

$$f(\mathscr{E},y)=k^{-1/2}W(\mathscr{E},y)+\mathrm{i}k^{1/2}W(\mathscr{E},-y) \tag{13}$$

where

$$k=(1+\mathrm{e}^{-2\pi\mathscr{E}})^{1/2}-\mathrm{e}^{-\pi\mathscr{E}} \tag{14}$$

$$k^{-1}=(1+\mathrm{e}^{-2\pi\mathscr{E}})^{1/2}+\mathrm{e}^{-\pi\mathscr{E}} \tag{15}$$

and the functions $W(\mathscr{E}, y)$, $W(\mathscr{E}, -y)$ have, respectively, the asymptotic forms [20, 23]

$$W(\mathscr{E}, y \to +\infty) \sim \sqrt{\frac{2k}{y}} \cos\left(\frac{y^2}{4} + \mathscr{E} \log y + \frac{\pi}{4} + \tfrac{1}{2}\Phi(\mathscr{E})\right) \tag{16}$$

and

$$W(\mathscr{E}, y \to -\infty) \sim \sqrt{\frac{2}{k|y|}} \sin\left(\frac{y^2}{4} + \mathscr{E} \log |y| + \frac{\pi}{4} + \tfrac{1}{2}\Phi(\mathscr{E})\right). \tag{17}$$

Therefore, the asymptotic forms of the solution (9), expressed in terms of the dimensionless variables y, $\mathscr{E}$, read

$$\psi(y \to +\infty, t) = \exp(\Lambda(t, y, \mathscr{E})) \sqrt{\frac{2}{y}} \exp\left[\mathrm{i}\left(\frac{y^2}{4} + \mathscr{E} \log y + \tfrac{1}{2}\Phi(\mathscr{E}) + \frac{\pi}{4}\right)\right] \tag{18}$$

$$\begin{aligned}\psi(y \to -\infty, t) = {} & \mathrm{i} \exp(\Lambda(t, y, \mathscr{E})) \sqrt{\frac{2}{|y|}} \Bigg\{[1 + \exp(-2\pi\mathscr{E})]^{1/2} \\ & \times \exp\left[-\mathrm{i}\left(\frac{y^2}{4} + \mathscr{E} \log|y| + \tfrac{1}{2}\Phi(\mathscr{E}) + \frac{\pi}{4}\right)\right] \\ & - \mathrm{e}^{-\pi\mathscr{E}} \exp\left[\mathrm{i}\left(\frac{y^2}{4} + \mathscr{E} \log|y| + \tfrac{1}{2}\Phi(\mathscr{E}) + \frac{\pi}{4}\right)\right]\Bigg\}\end{aligned} \tag{19}$$

where

$$\Lambda(t, y, \mathscr{E}) = -\frac{\mathrm{i}\gamma}{8\pi} y^2 + \frac{\gamma t}{4} - \mathrm{i}\mathscr{E}\Omega t. \tag{20}$$

We can now derive from (18) and (19) the reflection and transmission amplitudes according to the standard definitions. We get

$$R = \frac{-\mathrm{i}}{(2\pi)^{1/2}} \exp\left(-\pi^2 \frac{E}{\hbar\Omega}\right) \Gamma\left(\frac{1}{2} - \mathrm{i}\frac{E}{\hbar\Omega}\right) \tag{21}$$

and

$$T = \frac{1}{(2\pi)^{1/2}} \exp\left(-\pi^2 \frac{E}{\hbar\Omega}\right) \Gamma\left(\frac{1}{2} - \mathrm{i}\frac{E}{\hbar\Omega}\right) \tag{22}$$

where

$$\Gamma\left(\frac{1}{2} - \mathrm{i}\frac{E}{\hbar\Omega}\right) = \left[\frac{2\pi}{1 + \exp(-4\pi^2 E/\hbar\Omega)}\right]^{1/2} \exp(-\pi^2 E/\hbar\Omega) \exp(\mathrm{i}\Phi) \tag{23}$$

so that

$$\Phi(\mathscr{E}) = \arg \Gamma(\tfrac{1}{2} - \mathrm{i}\mathscr{E}). \tag{24}$$

Ford *et al* [23] have suggested a surprisingly simple yet accurate approximation to the phase shift, which is never wrong by more than about 2%, i.e.

$$\Phi(\mathscr{E}) \approx -\tfrac{1}{2}\,\mathscr{E}\ln\left[\left(\frac{\mathscr{E}}{e}\right)^2+\left(\frac{1}{4d}\right)^2\right] \tag{25}$$

where $(1/4d)^2 \approx 0.0197\ldots$.

3. Tunnelling times

From the knowledge of the phase shift as a function of the dimensionless energy parameter $\mathscr{E}$, we are able to deduce the explicit form of the main tunnelling times for the inverted CK equation.

The time delay (or, if negative, the time advance) is given by (3), or, in the dimensionless formalism

$$\tau_{\mathrm{d}} = \frac{1}{\Omega}\frac{\mathrm{d}\Phi}{\mathrm{d}\mathscr{E}}. \tag{26}$$

Therefore, we get from the approximate expression (25) of $\Phi(\mathscr{E})$

$$\tau_{\mathrm{d}} = -\frac{1}{\Omega}\left(0.5\ln X + \frac{(\mathscr{E}/e)^2}{X}\right) \tag{27}$$

where we put

$$X = \left(\frac{\mathscr{E}}{e}\right)^2 + \left(\frac{1}{4d}\right)^2. \tag{28}$$

Expression (27) of the time delay is an increasing function of the dissipation parameter γ (see figure 1); the case $\gamma = 0$ corresponds to the usual inverted harmonic oscillator. We thus recover the result that dissipation causes a decrease in the tunnelling process [10, 11, 25].

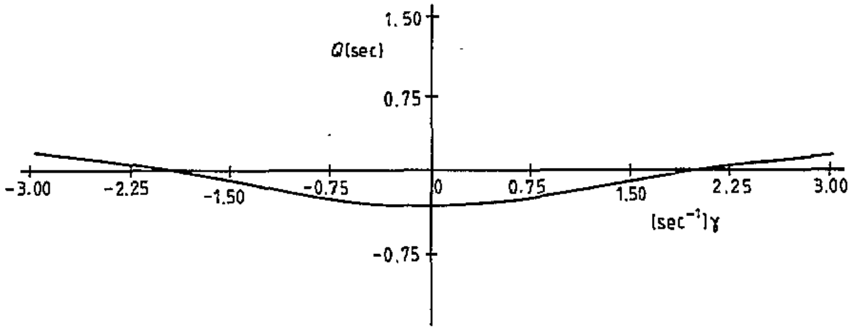

Figure 1. Plot of the time delay for $E = 1.5$ eV and $\omega = 1$ Hz ($\hbar = 1$).

A more accurate expression of the time delay is obtained by exploiting directly the form (24) of the phase shift, which can be written [25]

$$\Phi(\mathscr{E})=\frac{1}{2\mathrm{i}}\left[\log\Gamma(\tfrac{1}{2}- -\mathrm{i}\mathscr{E})-\log\Gamma(\tfrac{1}{2}+\mathrm{i}\mathscr{E})\right]. \tag{29}$$

Then, by (26), we get

$$\tau_{\mathrm{d}}=\frac{1}{\Omega}\frac{\mathrm{d}\Phi(\mathscr{E})}{d\mathscr{E}}=\frac{1}{\Omega}\operatorname{Re}\Psi\left(\frac{1}{2}+\mathrm{i}\mathscr{E}\right)=\frac{1}{\Omega}\operatorname{Re}\Psi\left(\frac{1}{2}+\mathrm{i}\frac{E}{\hbar\Omega}\right) \tag{30}$$

where

$$\Psi(z)=\frac{\mathrm{d}\log\Gamma(z)}{\mathrm{d}z}. \tag{31}$$

The duplication formula permits us to re-express (30) conveniently as [20, 26]

$$\tau_{\mathrm{d}}=\frac{1}{\Omega}\operatorname{Re}\left\{2\log 2+\Psi\left(1+\mathrm{i}\frac{E}{\hbar\Omega}\right)-2\Psi\left(1+2\mathrm{i}\frac{E}{\hbar\Omega}\right)\right\}. \tag{32}$$

It is useful to give the low-energy and the high-energy limits of the time delay (32). We have:

(i) for $|E|\ll\hbar\Omega$

$$\begin{aligned}\tau_{\mathrm{d}}&\approx\frac{1}{\Omega}\left\{2\log 2+C-7J(3)\frac{E^2}{\hbar^2\Omega^2}+\dots\right\}\\&\approx\frac{1}{\Omega}\left\{1.96-8.41\frac{E^2}{\hbar^2\Omega^2}+\dots\right\}\end{aligned} \tag{33}$$

where $C\approx 0.577$ is Euler's constant and $J(3)=\Sigma_{n=1}^{\infty}n^{-3}\approx 1.202$;

(ii) for $|E|\gg\hbar\Omega$

$$\tau_{\mathrm{d}}\approx\frac{1}{\Omega}\left\{-\log\left|\frac{E}{\hbar\Omega}\right|+\frac{1}{24}\frac{\hbar^2\Omega^2}{E^2}+\dots\right\}. \tag{34}$$

Let us also introduce the traversal time, defined by [20]

$$T(L,\mathscr{E})=2\log L+\frac{\mathrm{d}\Phi}{\mathrm{d}\mathscr{E}}=2\log L+\Omega\tau_{\mathrm{d}} \tag{35}$$

where

$$L=l\lambda^{-1} \tag{36}$$

$\lambda=(\hbar/2m\Omega)^{1/2}$ is the characteristic length of the inverted CK Hamiltonian and $(-l, l)$ is an interval which contains the barrier.

Then we can get the effective tunnelling velocity, $v_{\rm eff}$, related to $T(L, \mathscr{E})$ by

$$\begin{aligned} v_{\rm eff} &= \frac{2L}{T(L,\mathscr{E})} \\ &= L\left(\log L + \frac{1}{2}\frac{{\rm d}\Phi}{{\rm d}\mathscr{E}}\right)^{-1} \\ &= L\left(\log L + \frac{\Omega\tau_{\rm d}}{2}\right)^{-1}. \end{aligned} \tag{37}$$

4. Concluding remarks

Expressions (32)–(37) derived in the previous section permit us to consider, in the framework of the inverted CK model, some paradoxical aspects of the tunnelling phenomenon related to the particle speed in crossing the barrier. The first one is a paradox widely discussed in the literature [25, 27] and also present in the tunnelling behaviour under the barrier of the inverted harmonic oscillator [20]. Namely, for $E<0$, particles with lower energy travel faster, though of course with smaller probability, as we conjecture from (34) for $\gamma=0$. The same paradox in quantum tunnelling also occurs for Hamiltonian (2), but is even more enhanced. Indeed, in the presence of dissipation, the values of energy for which particles exhibit such a paradoxical behaviour are suppressed in the lower levels, i.e. particles with $|E| \gg \hbar\Omega$ (E<0) cross the barrier faster.

This is easily seen from (37): the effective speed $v_{\rm eff}$ becomes larger for particles with energy $|E| \gg \hbar\Omega$ (*or* $|E| \geqslant \hbar\omega$ for the inverted oscillator, $\gamma=0$), because ${\rm d}\Phi/{\rm d}\mathscr{E}$ (being a symmetric function of E) becomes more negative.

Moreover, for a sufficiently wide (or high) barrier, i.e. $L \gg 1$, or (see (36))

$$l\left(\frac{2m\Omega}{\hbar}\right)^{1/2} \gg 1 \tag{38}$$

the effective tunnelling velocity becomes arbitrarily large. This shows that the Hartmann–Fletcher effect [28, 19] also occurs for the dissipative tunnelling of the inverted CK oscillator (and, therefore, for the inverted oscillator in the absence of dissipation, i.e. for $\gamma=0$, $\Omega=\omega$).

At present, such an independence of the tunnelling speed from the barrier width is far from being understood on a clear physical basis [19]. However, let us recall that the Hartmann–Fletcher effect has just recently received experimental confirmation by the evidence for superluminal group velocities in the propagation of electromagnetic evanescent waves in waveguides [29] and in single-photon tunnelling [30]. In both cases, a phase-time approach is used. A possible explanation may be the presence of non-local effects (of electromagnetic and/or quantum nature) [31] in the phenomena considered. Due to the well known analogy between propagation of electromagnetic signals in waveguides and quantum tunnelling, it is therefore possible that our result on the existence of the Hartmann–Fletcher effect for the inverted CK model may find observational support from the introduction of dissipative effects in experiments similar to those reported in [29] and [30].

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